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(Editor)

DIFFERENTIAL GEOMETRY IN LORENTZ-MINKOWSKI SPACE

Proceedings of the Young Researcher Workshop
on Differential Geometry in Minkowski Space
Granada, Spain, April 17–20, 2017

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Preface

The contents of this book are the proceedings of the "Young Researcher Workshop on Differential Geometry in Minkowski Space", which held April 17-20, 2017, at the Instituto de Matemáticas (IEMath-GR), in Granada, Spain. The meeting was organized by the Department of Geometry and Topology of the University of Granada. The articles are the contributions by the speakers of the workshop, which present extended or modified versions of the lectures delivered at the meeting. This workshop brought together active young researchers on differential geometry in Lorentz-Minkowski space. On behalf of the organizers, we would like to thank all participants.

About twenty five participants from different countries attended in the meeting and discussed recent developments, specially in the theory of submanifolds in Lorentz-Minkowski space $\mathbb{L}^3$ and more general, in spacetimes. The topics considered coalesced around different themes, as for example, spacelike and timelike hypersurfaces, submanifolds with constant mean curvature and constant Gauss curvature, singularities of maximal surfaces in $\mathbb{L}^3$ and hypersurfaces invariant by isometries in Minkowski space. It is the hope of the editor that this volume will prove to be a stimulus for further research into this area.

The four invited speakers were Shintaro Akamine (Kyushu), Magdalena Caballero (Córdoba), Seher Kaya (Ankara) and Rahul Singh (Harish-Chandra). Shintaro Akamine focused on the diagonalizability of the shape operator of a regular timelike minimal surface, proving some relations between the symmetries of timelike minimal surfaces and the diagonalizability of the shape operator. Magdalena Caballero considered hypersurfaces that are both minimal in Euclidean space $\mathbb{E}^{n+1}$ and maximal in $\mathbb{L}^{n+1}$, showing that the level curves are minimal hypersurfaces in the n-dimensional Euclidean space. Seher Kaya told on the construction of examples of maximal surfaces in Lorentz-Minkowski space by using the Björling problem and studied the relation of duality between minimal and maximal rotational surfaces in terms of the Weierstrass-Enneper representation. Finally, Rahul Singh proved some new Ramanujan's type identities for complex numbers by using the Weierstrass-Enneper representation of maximal surfaces and he told on the singular Björling problem on maximal surfaces.

Besides the articles of the invited speakers, we highlight two articles. First, the survey "On the duality between CMC surfaces in $\mathbb{E}(\kappa,\tau)$ and $\mathbb{L}(\kappa,\tau)$", by José M. Manzano (King's College London), where he gives a uniform treatment of the duality between certain families of spacelike graphs with constant mean curvature in Riemannian and Lorentzaian homogeneous 3-manifolds with isometry group of dimension 4. The second paper is "Spacelike surfaces of constant Gaussian curvature in Lorentz-

Minkowski space" by Andrea Seppi (Pavia), which studies spacelike surfaces of constant Gaussian curvature K in the 3-dimensional Lorentz-Minkowski space showing that the space of entire K-surfaces with bounded second fundamental form is naturally parameterized, up to translations, by the tangent space at the trivial point of universal Teichmüller space.

The workshop was possible thanks to the financial support of different institutions. The main support came from the Research Project 'Geometric Analysis' (MT2014-52368-P) who supported the main speakers of the conference. Also other institutions of the University of Granada supported the event: the Department of Geometry and Topology, Plan Propio of the University of Granada and Doctorado de Matemáticas of the International Postgraduate School.

We also would like to express our thanks to the Instituto de Matemáticas (IEMath-GR) for letting us use their multimedia rooms and technical support.

Rafael López (Editor)
Granada, October 2017

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Contributions

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On the diagonalizability of the shape operator of timelike minimal surfaces in Lorentz-Minkowski 3-space

Shintaro Akamine

Abstract In this article we investigate the diagonalizability of the shape operator of regular timelike minimal surfaces in the Lorentz-Minkowski 3-space. First we review the results in [1], and by using them we give a characterization of flat B-scrolls, which gives another proof of the result by Clelland [4] for the case of timelike minimal surfaces. Moreover we also prove some relations between symmetries of timelike minimal surfaces and the diagonalizability of the shape operator.
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Keywords: Lorentz-Minkowski space, timelike minimal surface, diagonalizability of the shape operator, Gaussian curvature.

1 Introduction

A timelike (resp. spacelike) surface in Lorentz-Minkowski 3-space $\mathbb{L}^3$ with signature $(-,+,+)$ is a surface whose first fundamental form is a Lorentzian (resp. Riemannian) metric. Contrary to the case of spacelike surfaces, the shape operator of a timelike surface is not always diagonalizable over real number field $\mathbb{R}$. The diagonalizability of the shape operator is closely related to the mean curvature H and the Gaussian curvature K of a timelike surface because the characteristic equation of the shape operator of a timelike surface in $\mathbb{L}^3$ is written as $\lambda^2 - 2H\lambda + K = 0$. In this article we deal with surfaces whose mean curvature H vanishes identically, and in this case the diagonalizability of the shape operator is determined by the Gaussian curvature.

A spacelike surface in $\mathbb{L}^3$ whose mean curvature vanishes identically is called a *maximal surface*, and its Gaussian curvature K is always non-negative (see [11]). On the other hand, a timelike surface in $\mathbb{L}^3$ whose mean curvature vanishes identically is called a *timelike minimal surface*. The shape operator of a timelike minimal

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surface is diagonalizable over $\mathbb{R}$ on points where the Gaussian curvature is negative, diagonalizable over the complex number field $\mathbb{C}$ on points where the Gaussian curvature is positive and flat points consist of umbilic points and quasi-umbilic points (the definition of a quasi-umbilic point is given in Section 2).

Until now, there have been some studies focusing on the diagonalizability of the shape operator, that is, the sign of the Gaussian curvature of timelike minimal surfaces. Related to (conformal) Bernstein problem in $\mathbb{L}^3$, Magid [14] and Milnor [16] studied entire timelike minimal graphs in $\mathbb{L}^3$, and they also investigated the sign of the Gaussian curvature of such graphs. On the Gaussian curvature, they pointed out existence of entire timelike minimal graphs with negative Gaussian curvature and with positive Gaussian curvature. Therefore, even for entire timelike minimal graphs, the diagonalizability of the shape operator is not determined in general. Recently, the author also discussed in [1] the diagonalizability of the shape operator of timelike surfaces with or without singularities.

In this article we discuss the diagonalizability of the shape operator of regular timelike minimal surfaces. In Section 3, we first review the results on the diagonalizability of the shape operator for regular timelike minimal surfaces proved in [1], and as a corollary we give another proof of the characterization of totally quasi-umbilic timelike surfaces in $\mathbb{L}^3$ given by Clelland [4] for timelike minimal surfaces (Corollary 3.4). In Section 4, we discuss relations between symmetries of timelike minimal surfaces and the diagonalizability of the shape operator. We prove the following theorem:

Theorem 1.1. *If a timelike minimal surface is locally symmetric with respect to a spacelike or timelike plane, then away from umbilic points the shape operator of the surface is diagonalizable over* $\mathbb{R}$ *near the plane. On the other hand, if a timelike minimal surface contains a piece of the spacelike or timelike line, then away from umbilic points the shape operator of the surface is diagonalizable over* $\mathbb{C} \setminus \mathbb{R}$ *near the line.*

2 Preliminaries

In this section we give some notions and known results about timelike minimal surfaces and null curves. We refer [8, 13, 18] for a detailed description of timelike surfaces. The Lorentz-Minkowski 3-space $\mathbb{L}^3$ is the 3-dimensional vector space $\mathbb{R}^3$ with the Lorentzian metric $\langle\ ,\ \rangle = -dt^2 + dx^2 + dy^2$, where (t, x, y) are the canonical coordinates of $\mathbb{R}^3$.

Let Σ be a 2-dimensional manifold. An immersion $f : \Sigma \longrightarrow \mathbb{L}^3$ is called *timelike* (resp. *spacelike*) if its *first fundamental form* $\mathrm{I} = f^*\langle\ ,\ \rangle$ is a Lorentzian (resp. Riemannian) metric on Σ. For a spacelike or timelike immersion f and its unit normal vector field ν, the *shape operator* (or the *Weingarten map*) S and the *second fundamental form* II are defined as follows:

$$df(S(X)) = -\nabla_X \nu, \quad \mathrm{II}(X,Y) = \langle \nabla_{df(X)} df(Y), \nu\rangle,$$

where X and Y are vector fields on Σ, and ∇ is the Levi-Civita connection on $\mathbb{L}^3$. An eigenvalue of S is called a *principal curvature* of f. The *mean curvature* H and the *Gaussian curvature* K of f are defined as

$$H = \mathrm{tr}\,\mathrm{II}/2, \quad K = \varepsilon \det S,$$

where

$$\varepsilon = \langle \nu, \nu\rangle = \begin{cases} 1 & \text{if } f \text{ is timelike,} \\ -1 & \text{if } f \text{ is spacelike.} \end{cases}$$

For any spacelike surface in $\mathbb{L}^3$, the shape operator S is always diagonalizable over $\mathbb{R}$ and we can take real principal curvatures λ_1 and λ_2 of such surface. Therefore the discriminant of S satisfies the following inequality (see [13] for details)

$$H^2 - \varepsilon K = \left(\frac{\lambda_1 + \lambda_2}{2}\right)^2 - \lambda_1\lambda_2 = \left(\frac{\lambda_1 - \lambda_2}{2}\right)^2 \geq 0. \tag{1}$$

A spacelike surface in $\mathbb{L}^3$ whose mean curvature vanishes identically is called a *maximal surface*, and the Gaussian curvature K of the surface satisfies $K \geq 0$ by the inequality above. A point $p \in \Sigma$ of a timelike or spacelike immersion f is called an *umbilic point* if the second fundamental form II is a multiple of the first fundamental form I at p. Flat points of a maximal surface in $\mathbb{L}^3$ consist of umbilic points by (1).

On the other hand, the discriminant of the shape operator S for a timelike surface, which is written as $H^2 - K$, can be taken any real value, that is,

(i) S is diagonalizable over $\mathbb{R}$, in this case $H^2 - K \geq 0$ and the equality holds on umbilic points,

(ii) S is diagonalizable over $\mathbb{C} \setminus \mathbb{R}$, in this case $H^2 - K < 0$,

(iii) S is non-diagonalizable over $\mathbb{C}$, in this case $H^2 - K = 0$. A point satisfying this condition is called *quasi-umbilic point* ([4]).

A timelike surface in $\mathbb{L}^3$ whose mean curvature vanishes identically is called a *timelike minimal surface*, and flat points of a timelike minimal surface consist of umbilic points and quasi-umbilic points.

After a rigid motion in $\mathbb{L}^3$, any timelike surface can be locally expressed as the graph of a function $y = h(t, x)$. On the coordinate system (t, x), the first fundamental form is written as

$$\mathrm{I} = (-1 + h_t^2)dt^2 + 2h_t h_x dt dx + (1 + h_x^2)dx^2, \quad -1 + h_t^2 - h_x^2 < 0,$$

where the inequality comes from the timelike condition for the surface. The second fundamental form is written as

$$\mathrm{II} = \left(\frac{h_{tt}}{W}dt^2 + 2\frac{h_{tx}}{W}dtdx + \frac{h_{xx}}{W}dx^2\right), \quad W := \sqrt{1 - h_t^2 + h_x^2}.$$

Hence the mean curvature H is written as

$$H = \frac{(1 - h_t^2)h_{xx} + 2h_t h_x h_{tx} - (1 + h_x^2)h_{tt}}{2W^3},$$

and the Gaussian curvature K is written as

$$K = -\frac{h_{tt}h_{xx} - h_{tx}^2}{W^4}. \tag{2}$$

Here we note about a relation between the Gaussian curvatures with respect to the metrics induced from $\mathbb{E}^3$ and $\mathbb{L}^3$. The Gaussian curvature with respect to the induced metric from $\mathbb{E}^3$, we denote it by K_E, is

$$K_E = \frac{h_{tt}h_{xx} - h_{tx}^2}{W_E^4}, \quad W_E := \sqrt{1 + h_t^2 + h_x^2}.$$

Therefore, the relation $\mathrm{sgn}(K) = -\mathrm{sgn}(K_E)$ holds. This fact was noted in [10, Section 2], and the same relation is valid for spacelike surfaces, see [2, Section 4].

On the other hand, it is known that at each point of a timelike surface $f : \Sigma \longrightarrow \mathbb{L}^3$, there exists a local coordinate system (u, v) on which the first fundamental form can be written as $\mathrm{I} = 2F du dv$ with a non-zero function F ([18, page 13]). This local coordinate system is called a *null coordinate system*. A regular curve $\gamma : I \subset \mathbb{R} \longrightarrow \mathbb{L}^3$ satisfying $\langle \gamma', \gamma' \rangle = 0$ is called a *null curve*. A null coordinate system is a coordinate system on which the image of coordinate curves are null curves. On any null coordinate system (u, v), the first and second fundamental forms are written as

$$\mathrm{I} = 2F du dv, \quad \mathrm{II} = L du^2 + 2M du dv + N dv^2,$$

and hence the shape operator S and the mean curvature H are written as

$$S = \mathrm{I}^{-1}\mathrm{II} = \frac{1}{F}\begin{pmatrix} M & N \\ L & M \end{pmatrix}, \quad H = \frac{M}{F}.$$

The mean curvature H vanishes if and only if the coefficient $M = \langle f_{uv}, \nu \rangle$ vanishes. Since (u, v) is a null coordinate system, we have $f_{uv} \perp f_u,\ f_v$. Therefore, we obtain the well-known representation formula by McNertney [15]:

Proposition 2.1 ([15])**.** *Let $\varphi(u)$ and $\psi(v)$ be null curves in $\mathbb{L}^3$ such that $\varphi'(u)$ and $\psi'(v)$ are linearly independent for all u and v. Then*

$$f(u, v) = \frac{\varphi(u) + \psi(v)}{2} \tag{3}$$

is a timelike minimal surface. Conversely any timelike minimal surface can be written as (3) for some two null curves.

For a timelike minimal immersion, as classical minimal surface theory in Euclidean 3-space $\mathbb{E}^3$, there exists a one parameter family of isometric immersions and the conjugate surface as follows:

Definition 2.2 (cf. [8, 16, 18])**.** *The associated family of a timelike minimal immersion f written as (3) is a family $\{f_\lambda\}_{\lambda \in \mathbb{R}\setminus\{0\}}$ consists of*

$$f_\lambda(u, v) = \frac{\lambda\varphi(u) + \lambda^{-1}\psi(v)}{2},$$

and the conjugate surface $\hat{f}$ is

$$\hat{f}(u, v) = \frac{\varphi(u) - \psi(v)}{2}.$$

Remark 2.3. *As pointed out in [8], the conjugate surface $\hat{f}$ is not contained the associated family $\{f_\lambda\}_{\lambda \in \mathbb{R}\setminus\{0\}}$. The conjugate surface $\hat{f}$ is anti-isometric to f, that is, $\hat{f}$ satisfies*

$$\hat{f}^*\langle , \rangle = -f^*\langle , \rangle.$$

Example 2.4. *Let us take the null curve $\gamma(t) = (t, \cos t, \sin t)$ and $\varphi(u) = \gamma(u)$, $\psi(v) = \gamma(v)$. The timelike minimal surfaces*

$$f(u, v) = \frac{\varphi(u) + \psi(v)}{2}, \quad \hat{f}(u, v) = \frac{\varphi(u) - \psi(v)}{2}$$

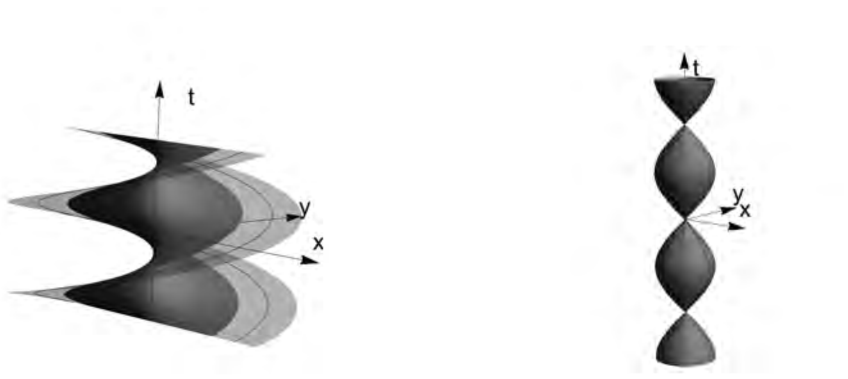

Figure 1: The timelike elliptic helicoid and the timelike elliptic catenoid.

are called the timelike elliptic helicoid and the timelike elliptic catenoid, respectively (see, for example, [9]). These surfaces have singular points, that is, points on which the maps are not immersed (see Figure 1). The elliptic helicoid f is an inner part of the usual helicoid in $\mathbb{E}^3$.

In the end of this section, we give notions of non-degeneracy and orientations of null curves.

Definition 2.5 (cf. [6, 17])**.** *A null curve $\gamma : I \subset \mathbb{R} \to \mathbb{L}^3$ is called non-degenerate (resp. degenerate) at $t \in I$ if γ' and γ'' are linearly independent (resp. dependent) at $t \in I$. A null curve $\gamma : I \to \mathbb{L}^3$ is called a non-degenerate null curve if γ is non-degenerate at every point.*

As pointed out in [17] or [1, Lemma 2.6], a null curve is non-degenerate at $t \in I$ if and only if $\det[\gamma'(t)\ \gamma''(t)\ \gamma'''(t)] \neq 0$. Hence we can define the notion of the orientation for non-degenerate null curves as follows:

Definition 2.6. *For a non-degenerate null curve γ, we define the orientation of γ by the sign of the* $\det[\gamma'\ \gamma''\ \gamma''']$.

The definition of the orientation of a non-degenerate null curve does not depend on the choice of parameters of the null curve. Let us take the following representation of a null curve γ (see also [16, Section 2])

$$\gamma(t) = \left(t - t_0, \int_{t_0}^{t} \cos A(\tau)d\tau, \int_{t_0}^{t} \sin A(\tau)d\tau\right), \quad A = A(t) \text{ is a function.}$$

For this parameter, we can compute $\det[\gamma'(t)\,\gamma''(t)\,\gamma'''(t)]$ as

$$\det[\gamma'(t)\,\gamma''(t)\,\gamma'''(t)] = (A'(t))^3. \tag{4}$$

Therefore the non-degenerate null curve γ has positive (resp. negative) orientation if γ' moves anticlockwise (resp. clockwise) on the lightcone

$$\mathbb{Q}^2 := \{v = (t, x, y) \in \mathbb{L}^3 \mid \langle v, v\rangle = 0,\, t \neq 0\}$$

as the time coordinate t increases.

3 Diagonalizability of the shape operator

In Section 2 we saw that flat points of a timelike minimal surface consist of umbilic points (flat points on which the shape operator is diagonalizable over $\mathbb{R}$) and quasi-umbilic points (flat points on which the shape operator is non-diagonalizable). In [1], the author gave characterizations of these flat points as follows:

Proposition 3.1 ([1]). *Let p be a point of a timelike minimal surface f, and φ and ψ be two generating null curves of f as in (3). Then the following statements hold.*

(i) *p is an umbilic point if and only if both φ and ψ are degenerate at p.*

(ii) *p is a quasi-umbilic point if and only if only one of φ or ψ is degenerate at p.*

It is well known that any totally umbilic timelike surface in $\mathbb{L}^3$ is an open set of a timelike plane ($H = 0$) or *pseudosphere* (*de-Sitter 2-space*) $S^2_1 = \{(t, x, y) \in \mathbb{L}^3 \mid -t^2 + x^2 + y^2 = r^2\}$ with radius $r > 0$ ($H = 1/r$) up to a rigid motion in $\mathbb{L}^3$ (see [13, Theorem 3.2], for example). On the other hand, Clelland [4] determined totally quasi-umbilic timelike surfaces in $\mathbb{L}^3$ as follows:

Theorem 3.2 ([4]). *Any totally quasi-umbilic timelike surface in $\mathbb{L}^3$ is a ruled surface with lightlike rulings.*

As a class of timelike ruled surfaces with a lightlike base curve and lightlike rulings, Graves [7] introduced the following class of ruled surfaces called *B-scrolls*.

Definition 3.3 ([7]). *Let $\alpha = \alpha(s)$ be a null curve and $\beta = \beta(s)$ be a lightlike vector field along α. Then a ruled surface $f(s, t) = \alpha(s) + t\beta(s)$ is called a B-scroll of α if* *there exists a frame* $\{A, B, C\}$ *such that vector fields* A, B *and* C *along* α *satisfying* $A = \alpha'$, $B = \beta$, $\langle A, B\rangle = \langle C, C\rangle = 1$, $\langle A, C\rangle = \langle B, C\rangle = 0$ *and*

$$\frac{d}{ds}(A, B, C) = (A, B, C)\begin{pmatrix} 0 & 0 & -b \\ 0 & 0 & -a \\ a & b & 0 \end{pmatrix},$$

where $a = a(s)$ *and* $b = b(s)$ *are functions along* α*.*

All B-scrolls and more generally all timelike ruled surfaces with lightlike rulings satisfy the condition $H^2 - K = 0$, see [5, Theorem 2] or [13, Example 3.5]. As a corollary of Proposition 3.1, we give another proof of Theorem 3.2 for timelike minimal surfaces and show that the surface becomes a flat B-scroll:

Corollary 3.4. *Any totally quasi-umbilic timelike minimal surface is a flat B-scroll.*

Proof. Let us consider a totally quasi-umbilic timelike minimal surface f and the null curve decomposition as in (3). By (ii) of Proposition 3.1, we may assume that $\psi = \psi(v)$ is degenerate at every point, and hence ψ becomes a straight line. Therefore, we can write the surface f as

$$f(u, v) = \frac{\varphi(u) + v\psi_0}{2}, \quad \psi_0 = (1, \cos\theta, \sin\theta), \tag{5}$$

where θ is a real constant. Next we prove the surface is a B-scroll. The following argument is based on [3, p. 25]. First we normalize the parameter u so that $\langle \varphi'(u), \psi_0\rangle = 4$. Let us take $A(u) = \varphi'(u)/2$ and $B = \psi_0/2$. Since

$$\langle A \times B, A \times B\rangle = -\langle A, A\rangle\langle B, B\rangle + \langle A, B\rangle^2 = 1,$$

we can take $C = A \times B$, where $\times$ is the Lorentzian vector product of $\mathbb{L}^3$. Since B is a constant vector, there exists a function $a = a(u)$ such that

$$\frac{d}{ds}(A, B, C) = (A, B, C)\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -a \\ a & 0 & 0 \end{pmatrix},$$

and hence f is a B-scroll of the non-degenerate null curve φ in (5). The flatness follows from the conditions $H^2 - K = 0$ and $H = 0$. □

Remark 3.5. *By the representation (5), we conclude that every totally quasi-umbilic timelike minimal surface in* $\mathbb{L}^3$ *is congruent with its conjugate one.*

About the diagonalizability of the shape operator away from flat points of a timelike minimal surface, the following theorem is known:

Theorem 3.6 (cf. [1, 16])**.** *Away from flat points of a timelike minimal surface $f : \Sigma \to \mathbb{L}^3$, the shape operator S is diagonalizable over $\mathbb{R}$ (resp. over $\mathbb{C} \setminus \mathbb{R}$) if and only if the two generating non-degenerate null curves φ and ψ in (3) have the same orientation (resp. different orientations).*

Based on the equation (4), Milnor [16] proved the above theorem by using Euclidean arclength parameters of the generating null curves φ and ψ. On the other hand, the author also obtained in [1] a stronger result than Theorem 3.6 which includes a construction method of conformal curvature line coordinates and conformal asymptotic coordinates by using pseudo-arclength parameters of φ and ψ.

Example 3.7. *Since the generating null curves of the timelike elliptic helicoid f in Example 2.4 have the same orientation, the shape operator is diagonalizable over $\mathbb{C} \setminus \mathbb{R}$ by Theorem 3.6. On the other hand, the generating null curves of the timelike elliptic catenoid $\hat{f}$ in Example 2.4 have different orientations. Therefore the shape operator is diagonalizable over $\mathbb{R}$.*

4 Symmetries and the diagonalizability of the shape operator

In this section we investigate relations between symmetries and the diagonalizability of the shape operator of timelike minimal surfaces. The following regular reflection principles were proved by Kim, Koh, Shin and Yang [9]:

Proposition 4.1 ([9])**.** *If a timelike minimal surface is perpendicular to a spacelike or timelike plane, then the surface is locally symmetric with respect to the plane. On the other hand, if a timelike minimal surface contains a piece of a spacelike or timelike line, then the surface is locally symmetric with respect to the line.*

They also proved a relation between the above reflection properties and taking the conjugate surface of a timelike minimal surface as follows:

Proposition 4.2 ([9])**.** *If a timelike minimal surface is perpendicular to a spacelike or timelike plane, then its conjugate surface contains a piece of a line normal to the plane. Conversely, if a timelike minimal surface contains a piece of a spacelike or timelike line, then its conjugate surface is perpendicular to a plane normal to the line.*

Since the conjugate surface of a timelike minimal surface is anti-isometric to the original surface, the signs of the Gaussian curvatures of these two surfaces are different. Hence the reflection principles as above imply that there are relations between symmetries and the diagonalizability of the shape operator of timelike minimal surfaces. Here we give a proof of Theorem 1.1 in Introduction.

Proof. [of Th. 1.1] In each case, we may assume that the surface is locally written as a graph over a domain on the tx-plane, we denote it by $y = h(t, x)$, after a rigid motion in $\mathbb{L}^3$. First we prove (i). Without loss of generality, we may assume that the surface is locally symmetric with respect to the xy-plane (resp. ty-plane), which is equivalent to the even symmetry of $h = h(t, x)$ with respect to the parameter t (resp. x). By the even symmetry with respect to t (resp. x), $h_t(0, x) = h_{tx}(0, x) = 0$ (resp. $h_x(t, 0) = h_{xt}(t, 0) = 0$) holds. Putting them into the zero mean curvature equation for h

$$(1 - h_t^2)h_{xx} + 2h_t h_x h_{tx} - (1 + h_x^2)h_{tt} = 0,$$

we have

$$h_{xx} = (1 + h_x^2)h_{tt} \text{ on } t = 0 \quad (\text{resp. } h_{tt} = (1 - h_t^2)h_{xx} \text{ on } x = 0). \tag{6}$$

By (6) and the timelike condition $-1 + h_t^2 - h_x^2 < 0$, h_{xx} and h_{xx} have the same sign on $t = 0$ (resp. $x = 0$). By (2) the Gaussian curvature K on $t = 0$ (resp. $x = 0$) is

$$K = -\frac{h_{tt}h_{xx}}{W^4}, \tag{7}$$

and hence away from flat points K is negative on $t = 0$ (resp. $x = 0$). Moreover, if there exists a flat point on $t = 0$ (resp. $x = 0$), then $h_{tt} = 0$ or $h_{xx} = 0$ on the point by (7). If h_{tt} vanishes at the point, h_{xx} also vanishes and vice versa by (6), and hence the shape operator S vanishes at the point. Therefore there is no quasi-umbilic point on $t = 0$ (resp. $x = 0$).

Next we prove (ii). Without loss of generality, we may assume that the surface is locally symmetric with respect to the x-axis (resp. t-axis), which is equivalent to the odd symmetry of $h = h(t, x)$ with respect to the parameter t (resp. x). By the odd symmetry with respect to t (resp. x), $h_{tt}(0, x) = h_{xx}(0, x) = 0$ (resp. $h_{tt}(t, 0) = h_{xx}(t, 0) = 0$) holds. By (2) the Gaussian curvature K on $t = 0$ (resp. $x = 0$) is

$$K = \frac{h_{tx}^2}{W^4}, \tag{8}$$

and hence away from flat points K is positive on $t = 0$ (resp. $x = 0$). Moreover, if there exists a flat point on $t = 0$ (resp. $x = 0$), then $h_{tx} = 0$ on the point by (8), which means that the shape operator S vanishes at the point. Therefore there is no quasi-umbilic point on $t = 0$ (resp. $x = 0$). □

Example 4.3. *The timelike elliptic catenoid and the timelike elliptic helicoid in Example 2.4 have the symmetry with respect to a timelike plane and the symmetry with respect to a spacelike line segment, respectively. This is another reason, which is different from that of Example 3.7, why these surfaces have negative Gaussian curvature and positive Gaussian curvature, respectively. We can also check similar symmetries for other rotational timelike minimal surfaces and ruled timelike minimal surfaces except flat B-scrolls (see [3], for example).*

Example 4.4 (cf. [1, 8, 12])**.** *If we take two null curves*

$$\varphi(u) = \frac{1}{2}(-u - \frac{u^3}{3}, u - \frac{u^3}{3}, u^2) \text{ and } \psi(v) = \frac{1}{2}(v + \frac{v^3}{3}, v - \frac{v^3}{3}, v^2).$$

The surface obtained by these two null curves as in the equation (3) is called the timelike Enneper surface of isothermic type or an analogue of Enneper's surface. These two null curves have negative and positive orientations, respectively. By Theorem 3.6, the shape operator is diagonalizable over $\mathbb{R}$, and its conjugate has diagonalizable shape operator over $\mathbb{C} \setminus \mathbb{R}$. Since the timelike Enneper surface has the planar symmetry with respect to the timelike ty-plane, and its conjugate has the symmetry with respect to the spacelike x-axis, the diagonalizability of the shape operator of these surfaces also follows from Theorem 1.1, see Figure 2.

Figure 2: The timelike Enneper surface with a planar symmetry and its conjugate with a line symmetry (the yellow parts represent the singular points).

Remark 4.5. *There are some kinds of singular points on timelike minimal surfaces in Examples 2.4 and 4.4. In [1], the author investigated relations between shapes of singularities and the diagonalizability of the shape operator of timelike minimal surfaces.*

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References

[1] S. Akamine, *Behavior of the Gaussian curvature of timelike minimal surfaces with singularities*, submitted, arXiv:1701.00238.

[2] A.L. Albujer and M. Caballero, Geometric properties of surfaces with the same mean curvature in $\mathbb{R}^3$ and $\mathbb{L}^3$, *J. Math. Anal. Appl.* **445** (2017) 1013–1024.

[3] R.M.B. Chaves, M.P. Dussan and M. Magid, Björling problem for timelike surfaces in the Lorentz-Minkowski space, *J. Math. Anal. Appl.* **377** (2011) 481–494.

[4] J.N. Clelland, Totally quasi-umbilic timelike surfaces in $\mathbb{R}^{1,2}$, *Asian J. Math.* **16** (2012) 189–208.

[5] F. Dillen and W. Kuñel, Ruled Weingarten surfaces in Minkowski 3-space, *Manuscripta Math.* **98** (1999) 307–320.

[6] S. Fujimori, Y.W. Kim, S.-E. Koh, W. Rossman, H. Shin, M. Umehara, K. Yamada and S.-D. Yang, Zero mean curvature surfaces in Lorentz-Minkowski 3-space and 2-dimensional fluid mechanics, *Math. J. Okayama Univ.* **57** (2015) 173–200.

[7] L. Graves, Codimension one isometric immersions between Lorentz spaces, *Trans. Amer. Math. Soc.* **252** (1979) 367–392.

[8] J. Inoguchi and M. Toda, Timelike minimal surfaces via loop groups, *Acta Appl. Math.* **63** (2004) 313–355.

[9] Y. W. Kim, S.-E. Koh, H. Shin and S.-D. Yang, Spacelike maximal surfaces, timelike minimal surfaces, and Björling representation formulae, *J. Korean Math. Soc.* **48** (2011) 1083–1100.

[10] S. Lin, Curvature restrictions on convex, timelike surfaces in Minkowski 3-space, *Proc. Amer. Math. Soc.*, **128** (5) (1999) 1459–1466.

[11] O. Kobayashi, Maximal surfaces in the 3-dimensional Minkowski space $\mathbb{L}^3$, *Tokyo J. Math.* **6** (1983) 297–309.

[12] J. Konderak, A Weierstrass representation theorem for Lorentz surfaces, *Complex Var. Theory Appl.* **50** (5) (2005) 319–332.

[13] R. López, Differential Geometry of curves and surfaces in Lorentz-Minkowski space, *Int. Electron. J. Geom.* **7** (2014) 44–107.

[14] M. Magid, The Bernstein problem for timelike surfaces, *Yokohama Math. J.* **37** (1989) 125–137.

[15] L. McNertney, *One-parameter families of surfaces with constant curvature in Lorentz 3-space*, Ph.D. thesis, Brown University (1980).

[16] T.K. Milnor, Entire timelike minimal surfaces in $\mathbb{E}^{3,1}$, *Michigan Math. J.* **37** (1990) 163–177.

[17] Z. Olszak, A note about the torsion of null curves in the 3-dimensional Minkowski spacetime and the Schwarzian derivative, *Filomat* **29** (2015) 553–561.

[18] T. Weinstein, *An Introduction to Lorentz Surfaces*, de Gruyter Exposition in Math. **22**, Walter de Gruyter, Berlin (1996).